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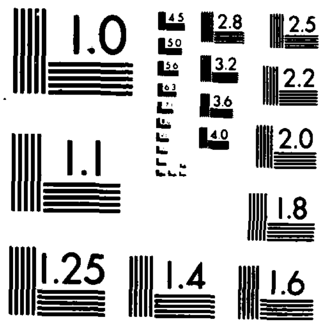

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Solution of the Algebraic Riccati Equation for Parabolic Systems

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Laurence R. Riddle and Howard L. Weinert

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Solution to the Algebraic Riccati Equation for Parabolic Systems *

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ABSTRACT

This paper presents an analytical solution to the operator algebraic Riccati equation (ARE) for selfadjoint parabolic systems. The solution to the operator ARE is important in the design of the steady-state, on-line filter for estimating the system's states. This analytical solution is derived by considering the operator analog of Potter's method of using the Hamiltonian system's eigenvectors and eigenvalues to solve a finite-dimensional ARE. As an example of using this analytical solution, the steady-state filtering error covariance for the 2-D heat equation is studied.

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1. Introduction

In this paper, we derive an analytical solution to the operator algebraic Riccati equation (ARE) on a Hilbert space H:

$$AP+PA^{*}+Q = PRP \tag{1.1}$$

when $-A$ is a strongly positive (coercive) selfadjoint operator that generates a continuous semigroup, Q is positive-definite, bounded and commutes with A, and R is bounded and nonnegative-definite. We also assume that the domain of A^2 is dense in H, and is contained in the domain of A, and that the positive square root of Q commutes with A. In the context of distributed parameter filtering and control, one needs a solution to Eq. (1.1) that is bounded, nonnegative, selfadjoint and that maps the domain of A into itself. Gibson [3] has shown that under the above assumptions, Eq. (1.1) has a unique solution with these properties.

An example where one would solve Eq.(1.1) with the above assumptions on A, Q and R is the design of a steady-state filter to estimate a process governed by the heat equation:

$$\frac{\partial}{\partial t}u(\mathbf{x},t) = \nabla^2 u(\mathbf{x},t)+\epsilon(\mathbf{x},t)\ ,\ \mathbf{x}\in\Omega\ \subset\ \mathbf{R}^3\ ,\ t\geq 0 \tag{1.2}$$

$$\frac{\partial}{\partial n}u(\mathbf{x},t) = 0\ ,\ \mathbf{x}\in\partial\Omega$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative, and the observations are

$$y_j(t) = \int_\Omega C_j(\mathbf{x})u(\mathbf{x},t)d\mathbf{x}+w_j(t)\ ,\ j = 1,\cdots,N$$

In Eq (1.2), $\epsilon(\mathbf{x},t)$ is a random process that is white in space and time with constant intensity Q, $w_j(t)$ is a scalar white noise process with intensity R^{-1}, and $A = \nabla^2$. Several applications using this model to describe the temperature distribution of heated metals have been reported [1],[9]. Numerical examples of calculating the steady-state filtering error covariance P for this model are given in [11].

2. Solving the ARE

In this section we will solve Eq. (1.1) by using the operator analog to the Potter method [10] of solving the ARE in finite dimensions. The Potter method of solving the ARE is summarized as follows: let the Hamiltonian matrix L be defined as

$$L = \begin{bmatrix} A & Q \\ R & -A^* \end{bmatrix}$$

and diagonalize L via

$$LM = M\Lambda$$

where

$$\Lambda = diag(\lambda_1, \cdots, \lambda_{2n}) \,,\, \lambda_1, \cdots, \lambda_n > 0$$

and

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

Then

$$P = M_{11} M_{21}^{-1}$$

is a nonnegative symmetric solution to the ARE , assuming M_{21} is invertible.

Using this approach in an operator setting, we will consider first a formal solution to the operator ARE. Let $Q = I$ and consider the following operator equation:

$$\begin{bmatrix} A & I \\ R & -A \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \lambda \tag{2.1}$$

where m_1, m_2 and λ are operators. It is easy to verify that the choices

$$m_1 = I \ , \ m_2 = -A + (A^2 + R)^{\frac{1}{2}}$$

$$\lambda = (A^2 + R)^{\frac{1}{2}}$$

satisfy Eq. (2.1) and thus

$$P_1 = (-A + (A^2 + R)^{\frac{1}{2}})^{-1} \tag{2.2}$$

satisfies Eq. (1.1) in a formal sense. If $Q \neq I$ then

$$P = S(-A + (A^2 + R)^{\frac{1}{2}})^{-1} S \tag{2.3}$$

satisfies Eq. (1.1), where $S^2 = Q$ and S is positive-definite.

To justify the formal solution given by Eq. (2.3) we need to show that P is bounded, nonnegative, selfadjoint, and maps the domain of A into itself. Since $-A > 0$ it follows that $A^2 + R$ has a unique positive square root that is selfadjoint and that the domain of A^2 is dense in the domain of $(A^2 + R)^{\frac{1}{2}}$[7, p. 281]. Therefore $-A + (A^2 + R)^{\frac{1}{2}}$ is strongly positive and selfadjoint with dense domain:

$$D(-A + (A^2 + R)^{\frac{1}{2}}) = D(-A) \ \cap D((A^2 + R)^{\frac{1}{2}}) \subset D(A^2) \subset D(A) \tag{2.4}$$

where $D(\cdot)$ denotes domain; so that P_1 as defined in Eq. (2.2) is bounded [4, p. 209]. Furthermore, P_1 is selfadjoint and nonnegative because $(-A + (A^2 + R)^{\frac{1}{2}})$ is [7, p. 272]. We will prove that P_1 maps the domain of A into the domain of

A.

The range of P_1 is contained in the domain of A by Eq. (2.4), and so it is sufficient to show that the domain of P_1 contains the domain of A. Since P_1 is bounded and necessarily closed (being selfadjoint), it follows [7, p. 269] that the domain of P_1 is H, and hence contains the domain of A.

Since we have assumed that the operator S commutes with A, if $x \in D(A)$ then $Sx \in D(A)$, hence P in Eq. (2.3) also maps $D(A)$ into $D(A)$. We have thus shown that the operator defined in Eq. (2.3) is the correct solution to Eq. (1.1) needed for filtering and control applications.

3. Numerical Considerations and Example

The analytical approach to solving the operator ARE presented in this chapter provides a computationally faster way of implementing the optimal gain for filtering and control applications. In our approach, finite-dimensional approximations to the operators in Eq (1.1) are done after the analytical solution has been obtained, resulting in a computational complexity of $O(6n^3)$, where n is the size of the matrix that approximates the operator A, whereas in other techniques [2] one approximates Eq (1.1) by a matrix ARE and then solves this equation using algorithms developed for the finite-dimensional case which require approximately $O(75n^3)$ operations [8]. We remark that a similar approach (with the same computational complexity $O(6n^3)$) to the finite-dimensional ARE has been considered [5],[6].

Another important difference between the analytical approach used in this chapter and those using a high order finite-dimensional approximation to Eq.

(1.1), is that the implementation of Eq. (2.3) calls for approximations of A and A^2 separately, whereas in approximating Eq. (1.1) one is implicitly using $\bar{A}^2$, where $\bar{A}$ is a finite-dimensional approximation to A, as an approximation to A^2. The fact that $\bar{A}^2$ may not be a good approximation to A^2 suggests that our approach may be more accurate.

The steady-state filtering error covariance for estimating the temperature profile of a heated square aluminum slab was calculated using the results obtained in the previous section. These calculations were based on the following model for the variation in temperature $u(x,y,t)$ above an assumed known ambient temperature T_0:

$$\frac{\partial}{\partial t}u(x,y,t) = \alpha\nabla^2 u(x,y,t)+\epsilon(x,y,t) \quad 0 \leq x \text{ , } y \leq L$$

with boundary conditions

$$u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,L,t) = 0$$

where

$$L = 1 meter \text{ , } \alpha = 5 \times 10^{-5} meter^2/second$$

and the observations are

$$y(t) = u(.5,.5,t)+w(t)$$

The input $\epsilon(x,y,t)$ is assumed to be white in space and time with intensity

$$Q = 1(degreeC)^2/(meter^2) \text{ } second$$

and the observation noise $w(t)$ is white with intensity

$$R^{-1} = 10^{-2}(degreeC)^2/second$$

In Eq. (2.3) ,a 10×10 finite-difference approximation to the operators A and

A^2 was used. Figure 1 shows a cross-section of the variances of the temperature variations $u(x,.5,t)$ and the filtering error $\tilde{u}(x,.5,t)$. It is evident from Figure 1 that the unobserved temperatures more that 10 cm away from the sensor are not being estimated very effectively. This result places some doubt on the possibility of estimating, with few sensors, the unobserved temperature distribution of heated metals in the presence of spatially white process noise.

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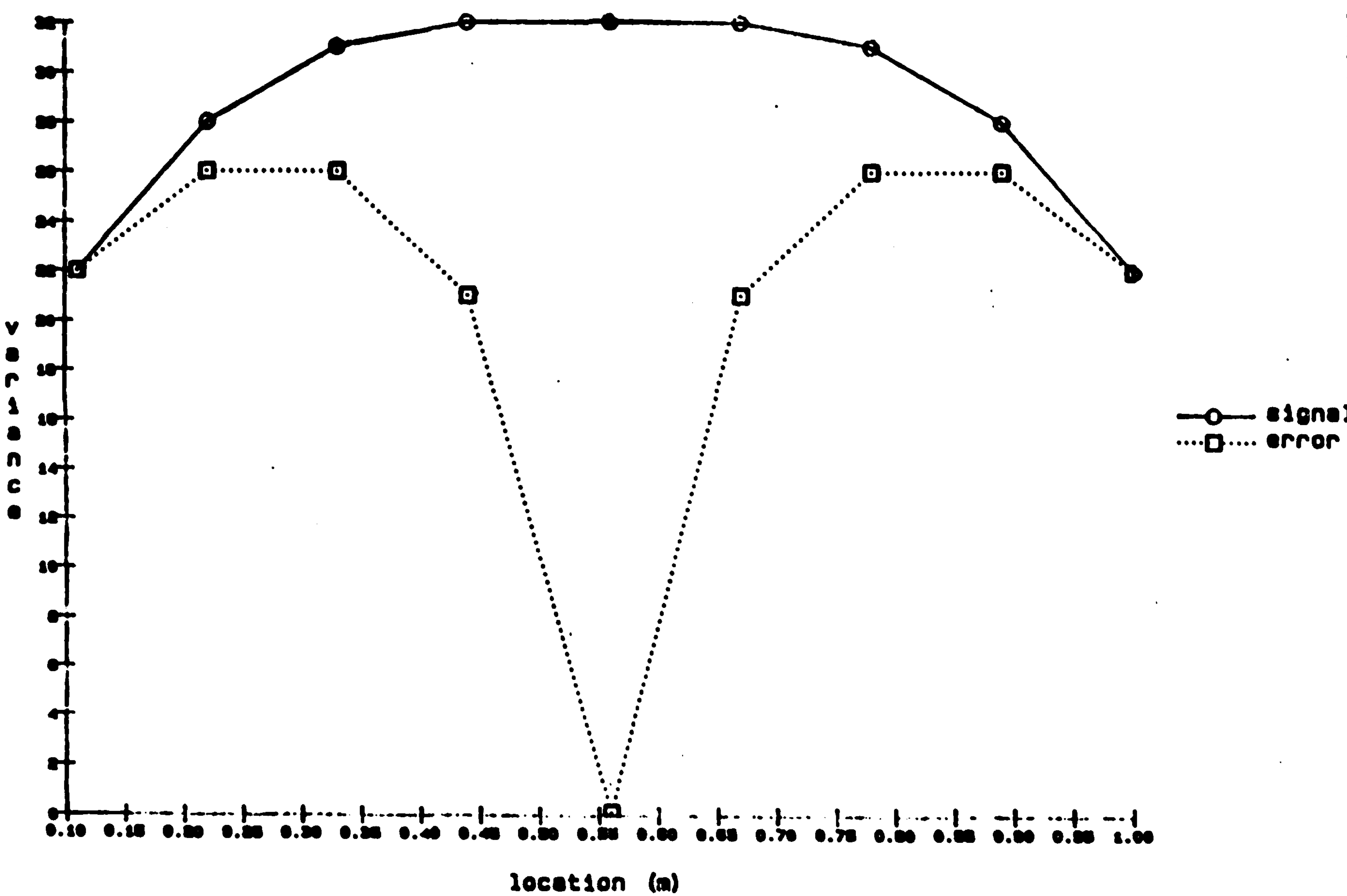
Figure 1

cross section of variances for 2-D Heat Eq



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